Markov effectuses and Riesz spaces

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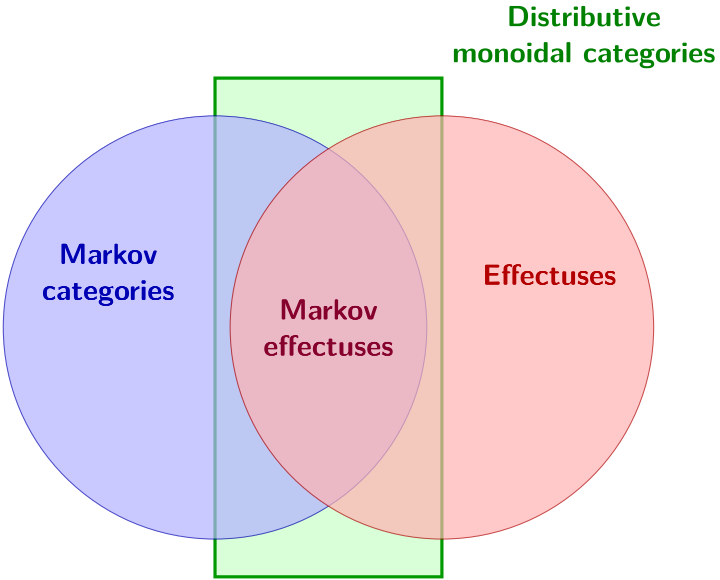
A landscape of channel-based categorical probability [1]



[1] ...may not be complete

A sketch of the effectus-theoretic approach

- Unbiased view of total and partial computation:

$$\mathbf{TotEf} \underset{\mathrm{Tot}}{\overset{\mathrm{Par}=\mathrm{Kl}(-+1)}{\rightleftarrows}} \mathbf{ParEf} \qquad (\simeq)$$

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- Example: $\mathrm{Kl}(\mathcal{D})$ and $\mathrm{Kl}(\mathcal{D}_{\leq 1})$.

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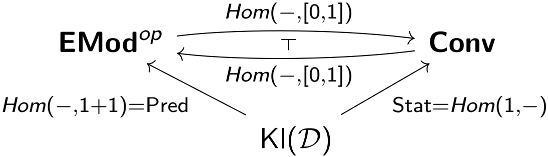
- Example: $\mathsf{Kl}(\mathcal{D})$ and $\mathsf{Kl}(\mathcal{D}_{\leq 1})$.
- Convention: write arrow $\rightarrow$ and $\circ$ for total maps, $\mathrel{\circ\!\!\rightarrow}$ and $\odot$ for partial maps.
- $f : X \rightarrow Y + 1$ is the same as $f : X \mathrel{\circ\!\!\rightarrow} Y$.

A sketch of the effectus-theoretic approach

Emphasis on coproducts, structure of (fuzzy) predicates $X \to 1 + 1$ and states $1 \to X$:



Total effectuses

A (total) effectus is a category **B** such that

- **B** has finite coproducts.
- **B** has a final object.
- The following squares are pullbacks in **B**.

$$
\begin{array}{ccc}
X + Y & \xrightarrow{\mathrm{id}+!} & X + 1 \\
{\scriptstyle !+\mathrm{id}}\downarrow & \lrcorner & \downarrow{\scriptstyle !+\mathrm{id}} \\
1 + Y & \xrightarrow[\mathrm{id}+!]{} & 1 + 1
\end{array}
\qquad\qquad
\begin{array}{ccc}
X & \xrightarrow{!} & 1 \\
{\scriptstyle \kappa_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \kappa_1} \\
X + Y & \xrightarrow[!+!]{} & 1 + 1
\end{array}
$$

- The following two maps are jointly monic in **B**.

$$
1 + 1 + 1 \underset{[\kappa_2,\kappa_1,\kappa_2]}{\overset{[\kappa_1,\kappa_2,\kappa_2]}{\rightrightarrows}} 1 + 1
$$

Understanding the axioms?

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- The left one allows for pairing of partial maps. That is, the coproduct $X+Y$ is a bit like a product in Par(**B**).

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$$\begin{array}{ccc} X+Y & \xrightarrow{\mathrm{id}+!} & X+1 \\ {\scriptstyle !+\mathrm{id}}\downarrow & \lrcorner & \downarrow{\scriptstyle !+\mathrm{id}} \\ 1+Y & \xrightarrow[\mathrm{id}+!]{} & 1+1 \end{array} \qquad \begin{array}{ccc} X & \xrightarrow{!} & 1 \\ {\scriptstyle \kappa_1}\downarrow & \lrcorner & \downarrow{\scriptstyle \kappa_1} \\ X+Y & \xrightarrow[!+!]{} & 1+1 \end{array}$$

- The left one allows for pairing of partial maps. That is, the coproduct $X+Y$ is a bit like a product in Par(**B**).
- The right one expresses some sort of zero-sum-freeness. E.g. $\mathrm{Kl}(\mathcal{D}_{\pm})$ fails this.

Understanding the axioms?

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$$1+1+1 \underset{[\kappa_2,\kappa_1,\kappa_2]}{\overset{[\kappa_1,\kappa_2,\kappa_2]}{\rightrightarrows}} 1+1$$

- **TotRel** fails this.
- Expresses something about cancellativity of addition of predicates $p : X \to 1+1$.

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- **TotRel** fails this.
- Expresses something about cancellativity of addition of predicates $p : X \to 1+1$.
- Makes the partial projections $\rhd_1 : X + Y \multimap X$ and $\rhd_2 : X + Y \multimap Y$ jointly monic in Par(**B**), where

$$\rhd_1 = [\mathrm{id}, \kappa_2] \qquad\qquad \rhd_2 = [\kappa_2, \mathrm{id}]$$

Examples of effectuses

Explanation	**Total**	**Partial**
(Partial) functions	**Sets**	**Par**
Discrete (sub)probability kernels	$\mathsf{Kl}(\mathcal{D})$	$\mathsf{Kl}(\mathcal{D}_{\leq 1})$
Measurable (sub)probability kernels	$\mathsf{Kl}(\mathcal{G})$	$\mathsf{Kl}(\mathcal{G}_{\leq 1})$
Dedekind σ-complete unital Riesz spaces with σ-normal positive (sub)unital maps	$\sigma\mathbf{URiesz}_{\sigma\mathbf{PU}}^{op}$	$\sigma\mathbf{URiesz}_{\sigma\mathbf{PSU}}^{op}$
C^*-algebras with positive (sub)unital maps	${\mathbf{CStar}_{\mathbf{PU}}}^{op}$	${\mathbf{CStar}_{\mathbf{PSU}}}^{op}$
Von Neumann algebras with completely positive, normal, (sub)unital maps	${\mathbf{vNA}_{\mathbf{CPNU}}}^{op}$	${\mathbf{vNA}_{\mathbf{CPNSU}}}^{op}$

Examples of Markov effectuses

Explanation	Total	Partial
(Partial) functions	**Sets**	**Par**
Discrete (sub)probability kernels	$\mathsf{Kl}(\mathcal{D})$	$\mathsf{Kl}(\mathcal{D}_{\leq 1})$
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The structure of partial maps (example)

- Let $f, g : X \to Y$ in $\mathsf{Kl}(\mathcal{D}_{\leq 1})$.
- The maps f and g are **compatible** (written $f \perp g$) if for all $x \in X$,

$$\sum_{y \in Y} f(y|x) + g(y|x) \leq 1.$$

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$$(f \ovee g)(y|x) = f(y|x) + g(y|x).$$

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$$(f \ovee g)(y|x) = f(y|x) + g(y|x).$$

- Clearly, this $\ovee$ is associative, and the zero map $\mathbf{0} : X \multimap Y$ is a neutral element.

The structure of partial maps (example)

- Composition preserves $\ovee$: PCM-enrichment

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- Composition preserves $\ovee$: PCM-enrichment
- Define the following order on maps $X \mathrel{\circ\!\!\rightarrow} Y$:

$$f \leq g \iff \forall x, y. f(y|x) \leq g(y|x).$$

- Composition preserves this order: poset-enrichment.

The structure of partial maps

Definition

A **partial commutative monoid** is a set X with a zero element $\mathbf{0}$, and partial binary operation $\oslash$ that satisfies the following, where we write $x \perp y$ if $x \oslash y$ is defined.

1. $x \perp \mathbf{0}$ and $x \oslash \mathbf{0} = x$.
2. If $x \perp y$ then $y \perp x$ and $x \oslash y = y \oslash x$.
3. If $x \perp y$ and $(x \oslash y) \perp z$, then $y \perp z$, $x \perp (y \oslash z)$, and $(x \oslash y) \oslash z = x \oslash (y \oslash z)$.

The structure of partial maps

Proposition

Let **B** be a total effectus.

1. Par(**B**) is enriched in PCMs.
2. Partial maps $X \multimap Y$ are ordered by:

$$f \leq g \iff \exists h. f \ovee h = g.$$

3. These orders form a poset-enrichment in $Par(\mathbf{B})$.

Predicates and scalars

- Maps $p : X \to 1 + 1$, equivalently maps $p : X \multimapdotinv 1$ are called **predicates**.
- In $\mathrm{Kl}(\mathcal{D})$, predicates correspond to fuzzy predicates $p : X \to [0, 1]$.

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- Maps $s : 1 \to 1 + 1$, equivalently maps $s : 1 \multimap 1$ are called **scalars**.
- In $\mathsf{Kl}(\mathcal{D})$, scalars correspond to elements of the unit interval $[0, 1]$.

The structure of predicates (example)

- There is a **truth** predicate $\mathbf{1} : X \to [0,1]$, given by $\mathbf{1}(x) = 1$.
- If $p : X \to [0,1]$, define its **orthocomplement** $p^{\perp} : X \to [0,1]$ by

$$p^{\perp}(x) = 1 - p(x).$$

Then $p \ovee p^{\perp} = 1$.

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Then $p \ovee p^\perp = 1$.

- Scalars $s, r \in [0,1]$ can be multiplied.
- Scalars in $[0,1]$ have an action on predicates $p : X \to [0,1]$, given by

$$(s \cdot p)(x) = s \cdot p(x).$$

The structure of predicates

Definition

An **effect algebra** is PCM $(X, \varovee, \mathbf{0})$ with an additional operation $(-)^{\perp} : X \to X$ that satisfies the following, where we write $\mathbf{1} = \mathbf{0}^{\perp}$.

1. $x^{\perp}$ is the unique element for which $x \varovee x^{\perp} = \mathbf{1}$.
2. If $x \perp \mathbf{1}$, then $x = 0$

The structure of predicates

Definition

An **effect monoid** is an effect algebra $(M, \ovee, \mathbf{0})$ with an additional binary operation & that satisfies the following.

1. & is associative, and has $\mathbf{1}$ as neutral element.
2. & preserves $\mathbf{0}$ and $\ovee$ in both arguments.

The effect monoid is called **commutative** if & is commutative.

The structure of predicates

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Definition

An **effect module** over an effect monoid M is a an effect algebra X with an M-action $- \cdot - : M \times X \to X$ that satisfies the following:

1. The action preserves $\mathbf{0}$ and $\ovee$ in both arguments.
2. $\mathbf{1}_M \cdot x = x$.
3. $(m \mathbin{\&} n) \cdot x = m \cdot (n \cdot x)$.

The structure of predicates

Proposition

Let **B** be an effectus.

1. The collection M of scalars $1 \to 1 + 1$ forms an effect monoid.
2. For all objects X, the collection $\mathrm{Pred}(X)$ of predicates $X \to 1 + 1$ is an effect module over M.

The predicate transformer functor

Proposition

Let $\mathbf{B}$ be an effectus. Write $\mathbf{EMod}_M$ for the category of effect modules over the scalars of $\mathbf{B}$, with module homomorphisms as maps.Then there is a functor $\mathrm{Pred} : \mathbf{B} \to \mathbf{EMod}_M^{op}$, defined on morphisms by

$$\mathrm{Pred}(f)(p) = p \circ f.$$

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- In $\text{Kl}(\mathcal{D})$, Pred is defined as follows:

$$\text{Pred}(X) = [0,1]^X$$

$$\text{Pred}(f : X \to \mathcal{D}(Y))(p)(x) = \sum_{y \in Y} f(y|x) \cdot p(y) = \mathop{\mathbb{E}}_{y \sim f(x)} p(y)$$

State-and-effect triangles (sketch)

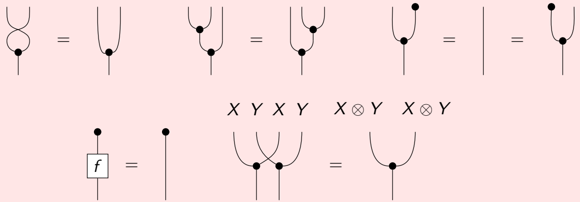
- States $1 \to X$ of an effectus form a convex set over the scalars.
- There is a 'state-and-effect triangle':

$$
\begin{array}{ccc}
\mathbf{EMod}_M^{op} & \underset{\textstyle\xleftarrow[\;Hom(-,M)\;]{}}{\overset{\textstyle\xrightarrow{\;Hom(-,M)\;}}{\top}} & \mathbf{Conv}_M \\
 & {\scriptstyle Hom(-,1+1)=\mathrm{Pred}} \nwarrow \quad \nearrow {\scriptstyle \mathrm{Stat}=Hom(1,-)} & \\
 & \mathbf{B} &
\end{array}
$$

Markov categories

Definition

A **Markov category** is a semicartesian symmetric monoidal category $(\mathbf{C}, \otimes, 1)$ with **copier** $\Delta_X : X \to X \otimes X$ and **discard** $!_X : X \to 1$ maps, such that



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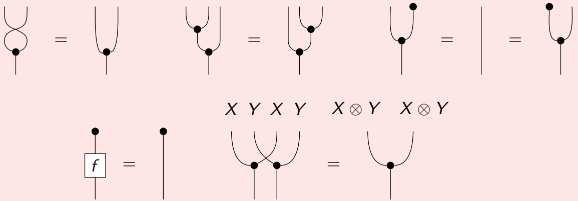
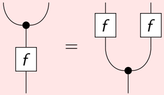


If $\mathbf{C}$ has coproducts, the Kleisli-category $\mathrm{Kl}(- + 1)$ is a **copy-discard category**.

Deterministic maps

Definition

A map $f : X \to Y$ in a Markov-category is called **deterministic** if

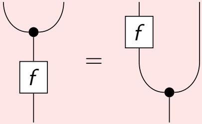


Markov effectuses

Definition

A **Markov effectus** is a category $\mathbf{B}$ that is both an effectus and a Markov category, and also satisfies the following:

1. The tensor preserves finite coproducts.
2. If $f : X \rightsquigarrow X$, such that $f \leq \mathrm{id}_X$, then the following holds in $Par(\mathbf{B})$:



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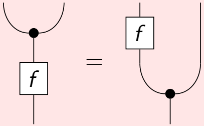

- Such an f is called **side-effect-free**.
- Write $\mathrm{End}_{\leq \mathrm{id}}(X)$ for the collection of side-effect-free maps on X.

The structure of predicates 2. (example)

- Predicates $[0,1]^X$ in $\mathrm{Kl}(\mathcal{D})$ form an effect monoid via pointwise multiplication.
- In $\mathrm{Kl}(\mathcal{D}_{\leq 1})$, side-effect-free maps $f \in \mathrm{End}_{\leq \mathrm{id}}(X)$ are of the following form for some $p \in [0,1]^X$:

$$f(x) = p(x)|x\rangle.$$

- This gives an isomorphism of effect monoids $Pred(X) \cong \mathrm{End}_{\leq \mathrm{id}}(X)$.

The structure of predicates 2.

Proposition

Let $\mathbf{B}$ a Markov effectus, X an object of $\mathbf{B}$.

1. Both $\mathrm{Pred}(X)$ and $\mathrm{End}_{\leq \mathrm{id}}(X)$ are commutative effect monoids.
2. $\mathrm{Pred}(X) \cong \mathrm{End}_{\leq \mathrm{id}}(X)$.

The structure of predicates 2.

Definition

Let M be an effect monoid. A **convex effect monoid** over M is an effect monoid that is also an effect module over M.

The structure of predicates 2.

Definition

Let M be an effect monoid. A **convex effect monoid** over M is an effect monoid that is also an effect module over M.

- Write **ConvEMon** for the category of convex effect monoids over M with maps that preserve both the effect module and effect monoid structure.
- Write $\mathbf{ConvEMon}_U$ for the category of convex effect monoids over M with maps that must preserve only the effect module structure.

The predicate transformer functor 2.

Proposition

Let $\mathbf{B}$ be a Markov effectus with scalars M.

1. The predicate functor $\mathrm{Pred} : \mathbf{B} \to \mathbf{EMod}_M$ restricts to a functor $\mathrm{Pred} : \mathbf{B} \to \mathbf{ConvEMon}_U$.
2. Write $\mathbf{B}_{det} \hookrightarrow \mathbf{B}$ for the subcategory of deterministic maps. Then the predicate functor fits in the commuting square below:

$$\begin{array}{ccc} \mathbf{ConvEMon} & \hookrightarrow & \mathbf{ConvEMon}_U \\ \uparrow & & \uparrow \scriptstyle{\mathrm{Pred}} \\ \mathbf{B}_{det} & \hookrightarrow & \mathbf{B} \end{array}$$

The predicate transformer functor 2.

Question

Is $\mathrm{Pred} : \mathbf{B} \to \mathbf{ConvEMon}_U$ monoidal?

The predicate transformer functor 2.

Question

Is Pred : $\mathbf{B} \to \mathbf{ConvEMon}_U$ monoidal?

Better question

Is $\mathbf{ConvEMon}_U$ even monoidal?

σ-effectuses

Definition

A σ-**partially additive monoid** is a set X with a partial countable sum operation $\bigovee$ that satisfies the following axioms, where we write $\perp U$ if a countable subset $U \subseteq X$ is summable.

1. Let $U \subseteq X$ be countable such that $U = \biguplus_{k\in\mathbb{N}} U_k$. Then $\perp U$ if and only if $\perp U_k$ for all k. Moreover, $\bigovee U = \bigovee_{k\in\mathbb{N}} \bigovee U_k$.
2. If $x \in X$, then $\perp\{x\}$ and $\bigovee\{x\} = x$.
3. $\perp U$ if and only if $\perp F$ for all finite subsets $F \subseteq U$.

σ-effectuses

Definition

An effectus $\mathbf{B}$ is a σ-**effectus** if

1. It has countable coproducts.
2. $Par(\mathbf{B})$ is σ-PAM-enriched.
3. *(Some technical axioms that are not important for the story.)*

Scalar division, real effectuses

Definition

An effect monoid M has **division** if for all $s, t \in M$ such that $s \leq t$ and $t \neq 0$, there is a unique $d \in M$ such that $d \,\&\, t = s$.

Scalar division, real effectuses

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Proposition (Cho, Westerbaan, van de Wetering, 2020)

Let **B** a σ-effectus whose scalars M have division. Then M is either $\{0\}$, $\{0, 1\}$, or $[0, 1]$.

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Let $\mathbf{B}$ a σ-effectus whose scalars M have division. Then M is either $\{0\}$, $\{0, 1\}$, or $[0, 1]$.

Definition

An effectus is called **real** if its scalars are the unit interval $[0, 1]$.

The structure of predicates 3.

Proposition (by me)

Let $\mathbf{B}$ a real Markov σ-effectus. Then predicates $\mathbf{B}(X, 1+1)$ form a lattice-ordered ω-complete convex effect monoid.

The predicate transformer functor 3.

Proposition (by me)

Let $\mathbf{B}$ be a real Markov σ-effectus.

1. The predicate functor $\mathrm{Pred} : \mathbf{B} \to \mathbf{ConvEMon}$ restricts to a functor $\mathrm{Pred} : \mathbf{B} \to \omega\mathbf{ConvEmon}_U$.
2. Write $\mathbf{B}_{det} \hookrightarrow \mathbf{B}$ for the subcategory of deterministic maps. Then the predicate functor fits in the commuting square below:

$$\begin{array}{ccc} \omega\mathbf{ConvEmon}_{\sigma}^{op} & \hookrightarrow & \omega\mathbf{ConvEmon}_{\sigma U}^{op} \\ \uparrow & & \uparrow \mathrm{Pred} \\ \mathbf{B}_{det} & \hookrightarrow & \mathbf{B} \end{array}$$

3. The predicate functor Pred is strong monoidal.

Faithfulness of the predicate transformer functor

Definition

A σ-effectus is called **predicate-separated** if any two $f, g : X \to Y$ satisfy $f = g$ whenever $p \circ f = p \circ g$ for all predicates $p : Y \to 1 + 1$.

Faithfulness of the predicate transformer functor

Definition

A σ-effectus is called **predicate-separated** if any two $f, g : X \to Y$ satisfy $f = g$ whenever $p \circ f = p \circ g$ for all predicates $p : Y \to 1 + 1$.

Proposition

A σ-effectus $\mathbf{B}$ is predicate -separated if and only if $\mathrm{Pred} : \mathbf{B} \to \omega\mathbf{ConvEmon}^{op}_{\sigma U}$ is faithful.

The predicate transformer functor 3.5. (sketch)

Proposition (by me)

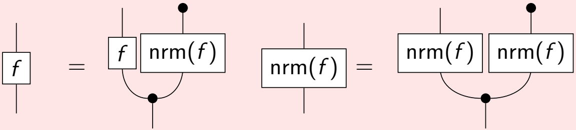
Let **B** be a real Markov σ-effectus. The predicate functor fits into the following diagram.

$$
\begin{array}{ccc}
\sigma\mathbf{CCstar}_{\sigma}^{op} & \hookrightarrow & \sigma\mathbf{CCstar}_{\sigma\mathbf{PU}^{op}} \\
\cong \big| & & \big\downarrow \cong \\
\sigma\mathbf{URiesz}_{\sigma}^{op} & \hookrightarrow & \sigma\mathbf{URiesz}_{\sigma\mathbf{PU}^{op}} \\
\cong \big| & & \big| \cong \\
\omega\mathbf{ConvEmon}_{\sigma}^{op} & \hookrightarrow & \omega\mathbf{ConvEmon}_{\sigma U}^{op} \\
\big\uparrow & & \big\uparrow \mathrm{Pred} \\
\mathbf{B}_{det} & \hookrightarrow & \mathbf{B}
\end{array}
$$

Normalisation

Proposition (by me)

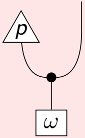
Let $\mathbf{C}$ be a Markov σ-effectus. Then every partial map $f : X \multimap Y$ admits a normalisation $\mathrm{nrm}(f) : X \multimap Y$ that satisfies the following in $Par(\mathbf{B})$:



Updating

Definition

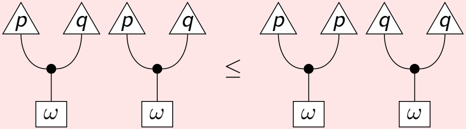
Let **B** be a Markov σ-effectus, $\omega : 1 \to X$, $p : X \to 1 + 1$. The **Bayesian update** $\omega|_p$ of the prior ω with evidence p is defined as the normalisation of the following.



Update increases validity

Proposition (by me)

1. Any real Markov σ-effectus $\mathbf{B}$ validates the **synthetic Cauchy–Schwarz inequality**. That is, for all $\omega : 1 \circ\!\!\!\rightarrow X$, and $p, q : X \circ\!\!\!\rightarrow 1$ the following holds in $Par(\mathbf{B})$.



2. For all $\sigma : 1 \to X$, and $p : X \to 1 + 1$ in $\mathbf{B}$, the following inequality holds:

$$p \circ \omega \leq p \circ \omega|_p.$$

Take-home message

- Markov effectuses are a very rich setting for categorical probability.
- My preferred setting: predicate-separated real Markov σ-effectuses
- Such categories all embed into categories of vector spaces/algebras via predicate transformation
- There are many equivalent characterisations of these predicate transformers

References

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