Categories of partial probabilistic computations

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Radboud Universiteit

Discrete probability distributions

Definition

A **discrete probability distribution** on a set X is a function $\varphi : X \to [0, 1]$ such that

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$$\frac{1}{3}\big|a, 1\big\rangle + \frac{1}{3}\big|b, 1\big\rangle + \frac{1}{3}\big|b, 3\big\rangle \quad \mapsto \quad \text{renormalise}$$

$$\frac{1}{2}\big|b, 1\big\rangle + \frac{1}{2}\big|b, 3\big\rangle \quad = \omega|_{\mathbb{A}}(b)$$

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???

Discrete probability subdistributions

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A **discrete probability subdistribution** on a set X is a function $\varphi : X \to [0,1]$ such that

1. $\mathrm{supp}(\varphi) = \{x \in X : \varphi(x) \neq 0\}$ is a finite set.
2. $\sum_{x \in X} \varphi(x) \leq 1$.

There is a monad $\mathcal{D}_{\leq} : \mathbf{Sets} \to \mathbf{Sets}$. Its Kleisli-category is symmetric monoidal.
Write $f : X \circ\!\!\to Y$ for a Kleisli-map $f : X \to \mathcal{D}_{\leq}(Y)$.

The structure of $\mathcal{K}l(\mathcal{D}_{\leq})$

- Copier maps:

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- Comparator maps:

$$\nabla_X : X \times X \multimap X$$
$$\nabla_X(x, y) = \begin{cases} 1|x\rangle & \text{if } x = y \\ \mathbf{0} & \text{otherwise} \end{cases}$$

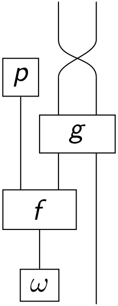
Copy-discard-compare (CDC) categories

Definition

A **copy-discard-compare category** is a symmetric monoidal category $(C, \otimes, I)$ with **copier** $\Delta_X : X \to X \otimes X$, **discard** $d_X : X \to I$, and **comparator** $\nabla_X : X \otimes X \to X$ maps such that...

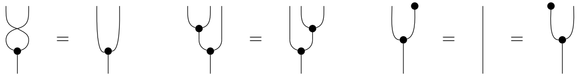
The anatomy of a string diagram

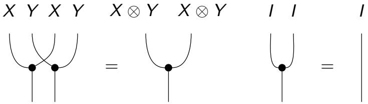
Let $(\mathbf{C}, \otimes, I, \alpha, \lambda, \rho, \sigma)$ be symmetric monoidal.



The axioms of copy-discard-compare (CDC) categories

- Copy and discard:





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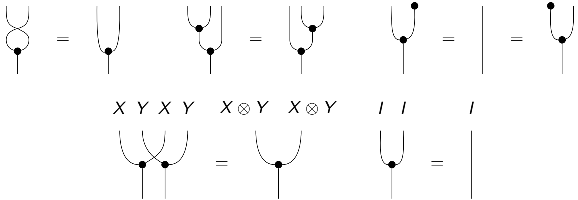
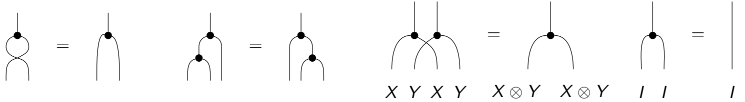


- Compare:



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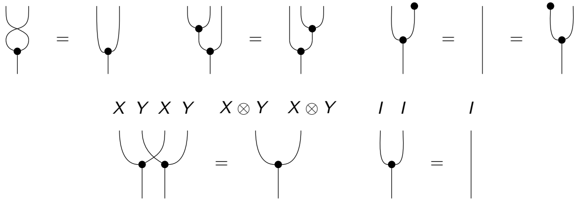

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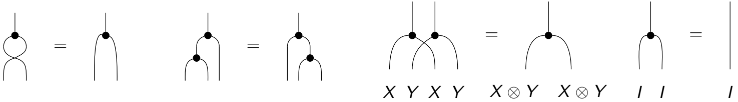
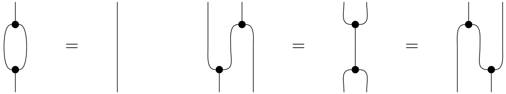


- Copy-compare interaction:



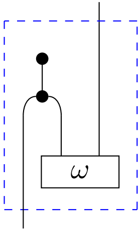
Examples of CDC-categories

- Sets and subprobability channels: $\mathcal{K}\ell(\mathcal{D}_{\leq})$
- Finite dimensional vector spaces and linear maps: **FinVect**
- Sets and relations: **Rel**
- Standard Borel spaces and subprobability kernels: $\textbf{BorelStoch}_{\leq}$
- ...

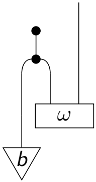
Our disintegration problem revisited

From $\omega : 1 \multimap \mathbb{A} \times \mathbb{N}$
extract $\omega|_{\mathbb{A}} : \mathbb{A} \multimap \mathbb{N}$:

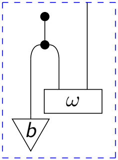
$\omega|_{\mathbb{A}} \quad :=$



Normalisation



$\mapsto$



$$\frac{1}{3}\Big|a,1\Big\rangle + \frac{1}{3}\Big|b,1\Big\rangle + \frac{1}{3}\Big|b,3\Big\rangle \quad\mapsto\quad \frac{1}{2}\Big|b,1\Big\rangle + \frac{1}{2}\Big|b,3\Big\rangle$$

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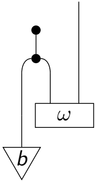

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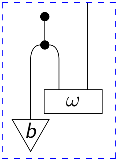

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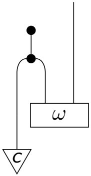

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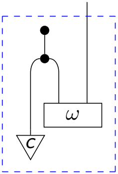

$$\tfrac{1}{3}\big|a,1\big\rangle + \tfrac{1}{3}\big|b,1\big\rangle + \tfrac{1}{3}\big|b,3\big\rangle \quad \mapsto \quad \mathbf{0}$$

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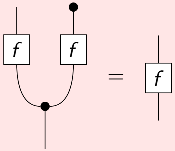
- The normalisation boxes select a least normalisation with respect to a partial order on normalisations.
- The dashed boxes enjoy many compositional properties.

Self-normalising maps

Definition

We call a map $f : X \to Y$ **self-normalising** if



For a subchannel $f : X \multimap Y$ in $\mathcal{K}l(\mathcal{D}_{\leq})$ this translates to

$$\forall x \in X. \sum_{y \in Y} f(x)(y) \in \{0, 1\}$$

The 'normalised by' relation

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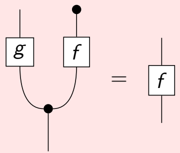
A map $g : X \to Y$ **normalises** $f : X \to Y$ if



In this case, we write $f \preceq g$.

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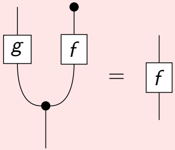

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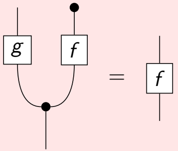

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- The dashed box should select the least normalisation.

Disintegration

'How to factorise $P(X, Y) = P(X) \cdot P(Y|X)$?'.

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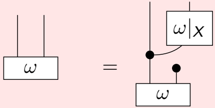
If $\omega : I \to X \otimes Y$, then a *disintegration* of ω is a map $\omega|_X : X \to Y$ that satisfies



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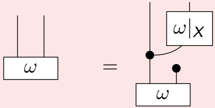

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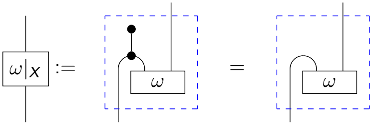
Can we use comparators and normalisation to compute a disintegration?



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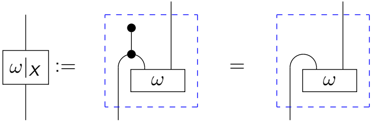

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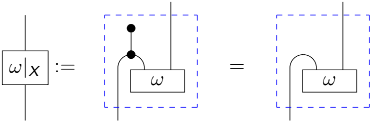

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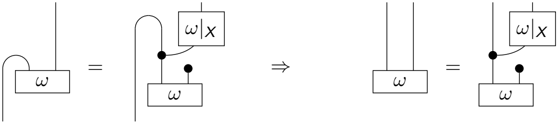

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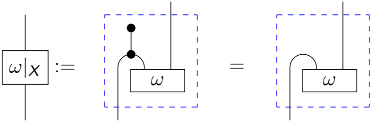

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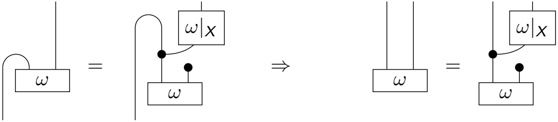

Remark

This does not recover disintegration in $\mathbf{BorelStoch}_{\leq}$

Outlook: notions of support

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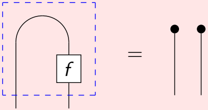

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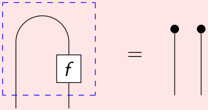

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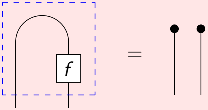

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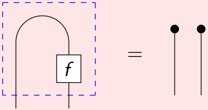

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- Everywhere full support maps are closed under many operations: composition, tensor, marginalisation, normalisation, disintegration.
- We use this notion for a compositional graphical calculus of disintegrations in a future article with Bart Jacobs, and Dario Stein.

Outlook: connections to effectus theory

- Effectus theory is a category theoretic framework for quantum and probability theory, focusing on the structure of well-behaved coproducts.

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- Effectuses need not have a monoidal structure. Adding that creates a very rich setting.
- Effectuses come in a total and partial flavour:

$$\mathbf{TotEf} \underset{\mathcal{T}ot}{\overset{\mathcal{K}\ell(-+1)}{\underset{\cong}{\rightleftarrows}}} \mathbf{ParEf}$$

References

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