String diagrams for probability

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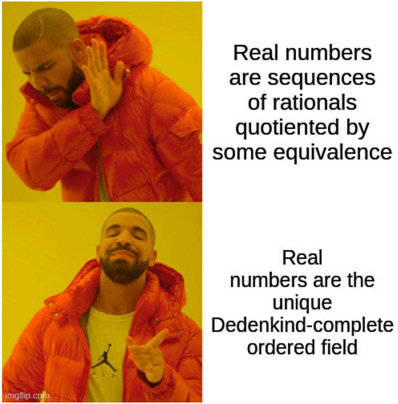
Mathematics: synthetic vs. analytic



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- Categorical probability comes with the convenient graphical calculus of string diagrams

I will demonstrate these aspects by an axiomatic study of equality comparison, disintegration, and normalisation of measures.

String diagrams for partial probabilistic computation: a motivating example

I roll a red and a blue die, and see that their sum is 10. What is the probability that the red die shows 5?

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$$\begin{aligned}
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&\mathit{sum} \leftarrow \mathit{blue} + \mathit{red} \\
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&\textbf{return } \mathit{red}
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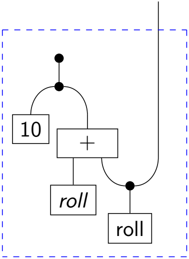
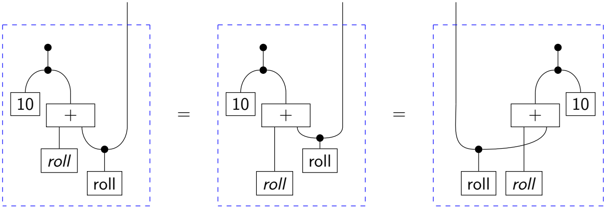


Why string diagrams?

- Topological transformations that 'make sense' are sound.



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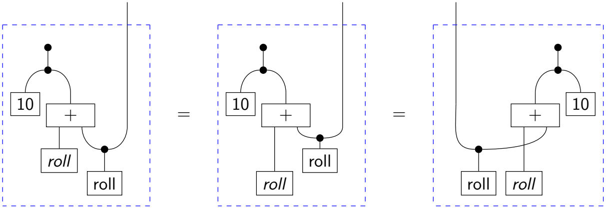
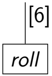


  This saves bookkeeping compared to a term language.
- Sometimes one needs terms with multiple outputs

The semantic universe: subprobabilistic computations

$$\mathit{roll}(0) : [1] \mathrel{\circ\!\!\!\rightarrow} [6]$$

$$\mathit{roll}(0) = \frac{1}{6}\big|1\big\rangle + \frac{1}{6}\big|2\big\rangle + \frac{1}{6}\big|3\big\rangle + \frac{1}{6}\big|4\big\rangle + \frac{1}{6}\big|5\big\rangle + \frac{1}{6}\big|6\big\rangle$$



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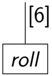
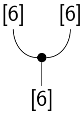


$$\mathrm{copy} : [6] \mathrel{\circ\!\!\!\rightarrow} [6] \times [6]$$

$$\mathrm{copy}(i) = 1|i, i\rangle$$



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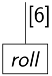

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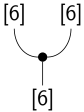
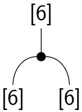


$$\text{compare} : [6] \times [6] \circ\!\!\!\rightarrow [6]$$

$$\text{compare}(i, j) = \begin{cases} 1|i\rangle & \text{if } i = j \\ \mathbf{0} & \text{otherwise} \end{cases}$$



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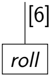

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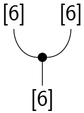

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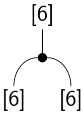
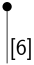


$$\text{discard} : [6] \circ\!\!\!\rightarrow [1]$$

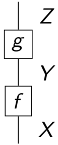
$$\text{discard}(i) = 1|0\rangle$$



Composing subprobability channels

- If $f : X \rightsquigarrow Y$, $g : Y \rightsquigarrow Z$, then

$$g \circ f : X \rightsquigarrow Z$$
$$(g \circ f)(z|x) = \sum_{y \in Y} f(y|x) \cdot g(z|y)$$



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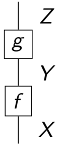
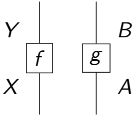


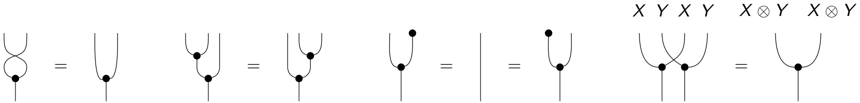
- If $f : X \mathrel{\circ\!\!\rightarrow} Y$, $g : A \mathrel{\circ\!\!\rightarrow} B$, then

$$(f \otimes g) : X \times A \mathrel{\circ\!\!\rightarrow} Y \times B$$
$$(f \otimes g)(y, b|x, a) = f(y|x) \cdot g(b|a)$$



The axioms of copy and compare (CDC-categories)

- Copy and discard:



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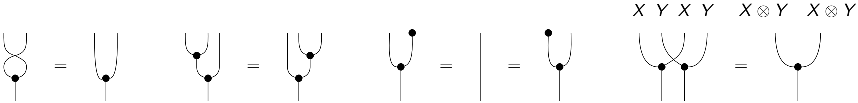
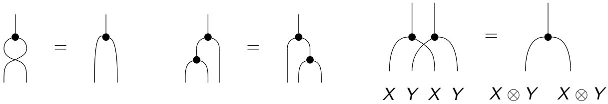


- Compare:



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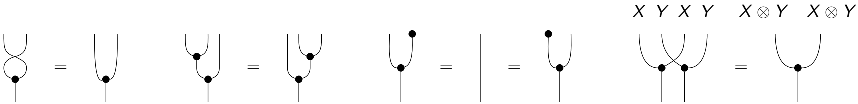

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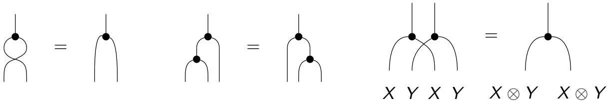
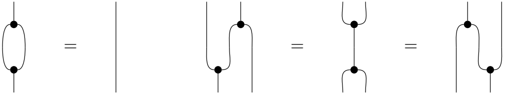


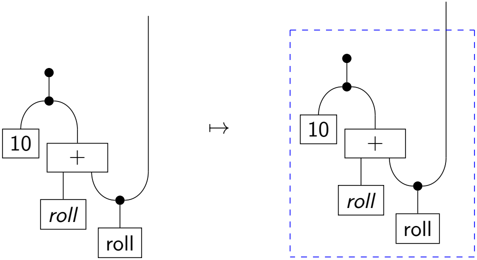
- Copy-compare interaction:



Examples of CDC-categories

- Finite sets and subprobability channels: **FinSubStoch**
- Finite dimensional vector spaces and linear maps: **FinVect**
- Sets and relations: **Rel**
- Standard Borel spaces and subprobability kernels: $\mathbf{BorelStoch}_{\leq}$
- ...

Normalisation



$$\tfrac{1}{36}\left|4\right\rangle + \tfrac{1}{36}\left|5\right\rangle + \tfrac{1}{36}\left|6\right\rangle \quad \mapsto \quad \tfrac{1}{3}\left|4\right\rangle + \tfrac{1}{3}\left|5\right\rangle + \tfrac{1}{3}\left|6\right\rangle$$

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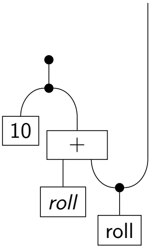

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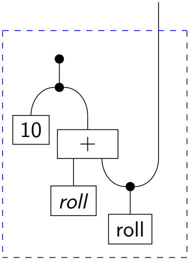

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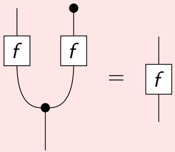
But how do we normalise the zero subdistribution?

Normalised maps

Definition

We call a map $f : X \multimap Y$ *normalised* if



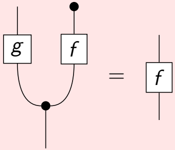
This translates to

$$\forall x \in X. \|f(x)\| \in \{0, 1\}$$

The ‘normalised by’ relation

Definition

A map $g : X \rightsquigarrow Y$ *normalises* $f : X \rightsquigarrow Y$ if



In this case, we write $f \preceq g$.

This translates to the following condition for all $x \in X$.

$$f(x) \neq \mathbf{0} \Longrightarrow \forall y \in Y . g(y|x) = \frac{f(y|x)}{\|f(x)\|}$$

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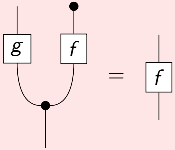

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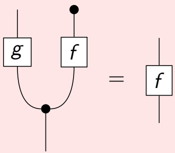

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- The relation $\preceq$ is a partial order on normalised maps
- The dashed box should select the least normalisation

The axioms of the dashed box

Definition

A *normalisation structure* assigns to every $f : X \multimap Y$ a normalised $\mathrm{nrm}(f) : X \multimap Y$ such that

1. $f \preceq \mathrm{nrm}(f)$

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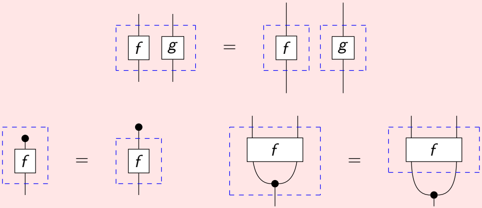
3. The following hold



The axioms of the dashed box

Proposition

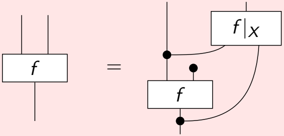
The dashed box assigns a least normalisation to each morphism.
Therefore, a CD-category can admit at most one normalisation structure.

Disintegration

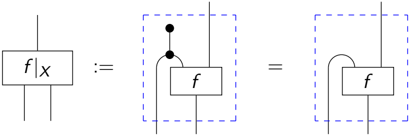
'How to compute $P(Y|X, Z)$ from $P(X, Y|Z)$?'.

Definition

If $f : Z \mathrel{\circ\!\!\!\to} X \times Y$, then a *disintegration* of f is a map $f_X : X \times Z \mathrel{\circ\!\!\!\to} Y$ that satisifes



Can we use comparator and normalisation to compute a disintegration?



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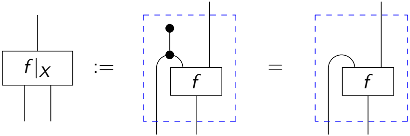

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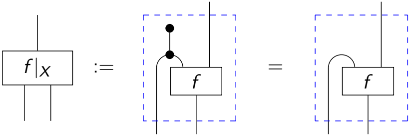

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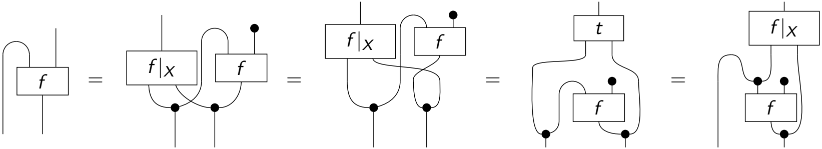

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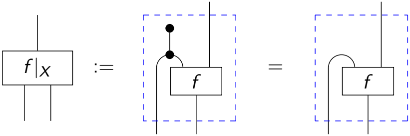

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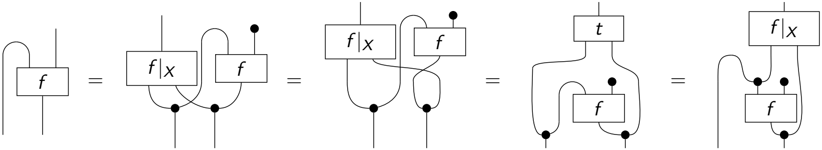


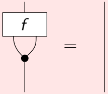
Remark

This does not recover disintegration in $\mathbf{BorelStoch}_{\leq}$

Comparators from disintegration

Proposition

A map $f : X \times X \circ\!\!\rightarrow X$ is a disintegration of the copier iff



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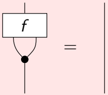
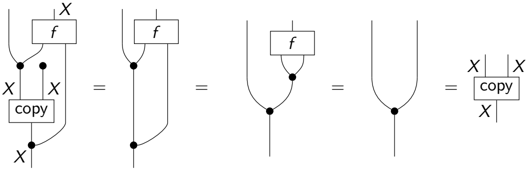


Proof:



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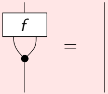

Proposition (by me)

A CD-category has comparators if and only if the copiers have minimal disintegrations.

Question to the room

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- The rewrite theory exists (see references)

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How to integrate string diagrammatic reasoning with proof assistants?

- There is a good attempt: ‘chyp’ by Aleks Kissinger. (https://github.com/akissinger/chyp)
- The rewrite theory exists (see references)
- It is not easy to integrate with existing proof assistant infrastructure (e.g. Coq)

References

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